

Differential Constraints in Chaotic Flows on Curved Manifolds

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Abstract

The Lagrangian derivatives of finite-time Lyapunov exponents and the corresponding characteristic directions are shown to satisfy time-asymptotic differential constraints in chaotic flows. The constraints are valid for any metric tensor, and are realised with exponential accuracy in time. Some of these constraints were derived previously for chaotic systems on low-dimensional Euclidean spaces, by requiring that the Riemann curvature tensor vanish in Lagrangian coordinates. The new derivation applies in any number of dimensions, predicts the number of constraints for a given flow, and provides a rigorous convergence rate of the constraints.

1 Introduction

A deep understanding of chaotic systems originates from the study of time-asymptotic dynamics [1]. The Lyapunov exponents (or characteristic exponents) are the most well-known instance; they describe the average rate of exponential separation of neighbouring trajectories, in the infinite-time limit [2–4]. The Lyapunov exponents are often used as a criterion for chaos: on a bounded domain, at least one positive exponent is required. They have proven quite rich, for example in characterising the fractal dimension of attractors in dissipative systems [5,6].

The convergence of the Lyapunov exponents is extremely slow, typically logarithmic in time. This is due to the fact that they are inherently nonlocal and require the fiducial trajectory (the trajectory along which the exponents are evaluated) to explore the attractor (or invariant region, for nonattracting systems) very thoroughly, as the greatest stretching often comes from rare

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excursions near hyperbolic fixed points. It is thus debatable whether the exponents are relevant to the finite-time dynamics of a single trajectory, or to an ensemble of tightly bunched trajectories, since a laboratory or numerical experiment will usually have ended long before the exponents have converged.

Hence the shift of emphasis to the finite-time Lyapunov exponents (FTLEs). These are the same as the Lyapunov exponents (*i.e.*, the infinite-time Lyapunov exponents) but averaged over a finite time, as the name implies. On short to moderate time scales, the FTLEs do not show convergence at all (that is, the average rate of separation is not exponential), but they are still connected to a rapid, quasi-exponential average separation of neighbouring trajectories.

The rapid separation of trajectories leads to a quasi-exponential convergence of the characteristic directions, the directions in the space of initial conditions (Lagrangian coordinates, which label fluid elements) exhibiting a given characteristic rate of separation. Thus, even though the FTLEs have not usually converged, on moderate time scales the characteristic directions converge very rapidly [7]. This is because the characteristic directions are local quantities, and they will typically converge after the fiducial trajectory has explored a portion of their neighbourhood.

In this paper we show that there are constraints on the derivatives of FTLEs and of characteristic directions. The derivatives are taken with respect to the initial starting coordinate of the fiducial trajectory—the Lagrangian coordinate—and so are called Lagrangian derivatives. The differential constraints are asymptotic, in the sense that they are satisfied with quasi-exponential accuracy in time.

For two- and three-dimensional dynamical systems, these constraints were first derived by Tang and Boozer [8] and Thiffeault and Boozer [9]. The method used there was to transform the Euclidean metric defined on the space in which the motion takes place (Eulerian coordinates) to the space of initial conditions (Lagrangian coordinates, or fluid labels). The metric in Lagrangian coordinates exhibits quasi-exponential blowup, but its associated Riemann curvature tensor must vanish identically, since the underlying space is flat. Thus, by balancing the various terms in the Riemann curvature tensor, it was shown that the curvature can only vanish identically if the FTLEs and characteristic directions obey asymptotic differential constraints. These constraints were also confirmed numerically for discrete maps [8,10] and for flows [9,11].

Here we propose a different approach, with an emphasis on flows. By performing a singular value decomposition (SVD) of the tangent map of the flow (the Jacobian of the transformation from Eulerian to Lagrangian coordinates), the quasi-exponentially growing terms can be evolved independently

from each other, so that their different growth rates do not pose a problem numerically. This amounts to a reorthonormalisation scheme, where the dominant eigenvectors are subtracted from the subdominant ones [12,7,13]. Greene and Kim [14,15] devised a numerical scheme to evolve the components of the SVD continuously. Thiffeault [11] extended their method to compute the Lagrangian derivatives of FTLEs and characteristic eigenvectors. This amounts to a numerical scheme for calculating the Hessian (the tensor of second partial derivatives) of the transformation from Eulerian (fixed or laboratory coordinates) to Lagrangian (fluid element labels). Related work on maps was done by Dressler and Framer [16] and Taylor [17]. Whereas the Jacobian contains information on the deformation of an infinitesimal ball of initial conditions into an ellipsoid, the Hessian describes higher-order (nonlinear) deformations.

The constraints are obtained in two steps. First, the time-asymptotic behaviour of Lagrangian derivatives is derived by analysing the equations for the components of the SVD, in a manner analogous to Ref. [11] but allowing for a nontrivial (*i.e.*, non-Euclidean) metric. Second, the symmetry of the Hessian is used, and Eulerian quantities are replaced by their asymptotic form. What is left are expressions generalising the constraints of Ref. [9] to an arbitrary number of dimensions and choice of metric. The new method of derivation also predicts the convergence rate of the constraints, something that was missing from earlier analyses.

Note that even though it is well-known that the Lyapunov exponents are independent of the particular positive-definite metric chosen to measure them [2,15], this is not true of the finite-time exponents and the characteristic directions. The details of the metric may affect the short- to moderate-time behaviour of the FTLEs greatly, and the characteristic directions are different even in the infinite-time limit. Hence, the derivation of the present results for the case of an arbitrary metric is relevant from a mathematical standpoint.

From a physical perspective, metrics that differ from the ordinary Euclidean case often arise. Examples are the advection–diffusion equation with an anisotropic, inhomogeneous diffusion tensor, and two-dimensional flow on a deformable membrane [18,19]. Even when the underlying space is Euclidean, there are many cases where it is advantageous to use coordinates better suited to the problem at hand, which lead to a nontrivial metric (spherical, toroidal, rotating, etc.). Hence, there is also physical motivation in keeping the derivation of the constraints as general as possible by considering an arbitrary, time- and space-dependent metric.

The outline of this paper is as follows. In Section 2 we review the mathematical description of a continuous dynamical system (flow) on a manifold with curvature. We define the Jacobian and the Hessian of the flow, and describe the metric on the Eulerian and Lagrangian tangent spaces of the manifold.

We also choose a convenient covariant time derivative. In Section 3 we define the orthonormal frames that will allow the separation of the different scales of the system along characteristic directions. This leads to the singular value decomposition and the time evolution equations for the orthonormal frames. From these we can define the finite-time Lyapunov exponents and characteristic directions. We devote Section 4 to showing that the nontrivial metric does not significantly affect the asymptotic forms of the Lagrangian derivatives derived in Ref. [11], allowing us to make use of the results therein. Section 5 contains the main results of the paper: we derive two types of constraints, and give the number of constraints of each type as a function of dimension and number of contracting directions (negative Lyapunov exponents). In Section 6, we make a connexion to the previous method (the curvature method of Refs. [8,9]) of finding constraints. Finally, we offer a summary and discussion of the implications of the results of the paper in Section 7.

2 Chaotic Dynamics in a Curved Space

In this section we introduce notation and review a number of concepts necessary for the remainder of the paper. In Section 2.1 we discuss a general dynamical system defined on a manifold. In Section 2.2 we equip that manifold with a metric tensor. To assist in writing equations in manifestly covariant form, in Section 2.3 we introduce a covariant form of the time derivative.

2.1 Dynamical Systems on a Manifold

We consider the dynamical system on a smooth compact manifold \mathcal{U}

$$\dot{x} = v(t, x), \quad (2.1)$$

where the overdot indicates a derivative with respect to time. Here $x = x(t) \in \mathcal{U}$ is a smooth function of $t \in \mathbb{R}$, and $v : \mathcal{U} \rightarrow T\mathcal{U}$ is a smooth vector field, with $T\mathcal{U}$ the tangent bundle of \mathcal{U} . We say that v generates a flow $\varphi^t : \mathcal{U} \rightarrow \mathcal{U}$, where $\varphi^t(a)$ is a diffeomorphism defined for all a in \mathcal{U} and $t \in \mathbb{R}$. The flow φ^t exists, and it satisfies Eq. (2.1) if we set $x = \varphi^t(a)$ [20, p. 304]. We choose φ^t such that $\varphi^0(a) = a$, defining a as the initial condition of a trajectory. By analogy with fluids, the x are called Eulerian coordinates and the a Lagrangian coordinates.

The derivative (also known as the tangent map, Jacobian, or push-forward) at a of φ^t is denoted $\varphi_{*a}^t : T_a\mathcal{U} \rightarrow T_{\varphi^t(a)}\mathcal{U}$, and satisfies the equation

$$\dot{\varphi}_{*a}^t = Dv(t, \varphi^t(a)) \cdot \varphi_{*a}^t, \quad \varphi_{*a}^0 = \text{Id}, \quad (2.2)$$

obtained by differentiating Eq. (2.1). We use a dot, as on the right-hand side of Eq. (2.2), to indicate the action of a linear map. In a particular chart, Eq. (2.2) can be written in coordinate form as

$$(\dot{\varphi}_{*a}^t)_q^i = \frac{\partial v^i}{\partial x^k} (\varphi_{*a}^t)_q^k, \quad (\varphi_{*a}^0)_q^i = \delta_q^i, \quad (2.3)$$

where $\partial v^i / \partial x^k$ is evaluated at $x = \varphi^t(a)$, and we are using the Einstein convention of summing over repeated indices from 1 to n , where n is the dimension of \mathcal{U} .

Equation (2.3) introduces a convention we shall use for the remainder of the paper: the indices i, j, k, ℓ denote vectors in Eulerian space ($T_x \mathcal{U}$), while the indices p, q, r denote vectors in Lagrangian space ($T_a \mathcal{U}$). Since φ_{*a}^t is a map from $T_a \mathcal{U}$ to $T_x \mathcal{U}$, we write its components as $(\varphi_{*a}^t)_q^k$.

If we imagine an infinitesimally small “ball” of points distributed around the initial condition a at $t = 0$, then the tangent map φ_{*a}^t captures the deformation of that ball into an ellipsoid under the action of the flow φ^t (see Ref. [13] for a more detailed description). For a chaotic flow, at least one axis of this ellipsoid grows exponentially in time, the growth rate being characterised by the largest Lyapunov exponent [2,1] (for finite times the growth rate is only roughly exponential, or quasi-exponential, but from now on we drop the quasi- prefix). This renders the direct numerical evaluation of Eq. (2.3) for moderately large times all but impossible due to roundoff error: all the columns of $(\varphi_{*a}^t)_q^i$ become aligned with the eigendirection of fastest stretching. A suitable matrix decomposition is needed to separate the growth of the different axes of the ellipsoid such that the fastest expanding direction does not overwhelm the others, that is, a reorthonormalisation scheme [12,7,13]. In Section 3.1 we achieve this using the singular value decomposition (SVD), in a manner analogous to Greene & Kim [14,15].

In this paper, as in earlier work [11], we are interested in deformations of a ball of initial conditions beyond ellipsoidal, so we want to obtain higher derivatives of φ^t . The second derivative of φ^t may be regarded as a symmetric bilinear form $\varphi_{**a}^t : T_a \mathcal{U} \times T_a \mathcal{U} \rightarrow T_{\varphi^t(a)} \mathcal{U}$, satisfying

$$\dot{\varphi}_{**a}^t = Dv(t, \varphi^t(a)) \cdot \varphi_{**a}^t + D^2v(t, \varphi^t(a))[\varphi_{*a}^t \otimes \varphi_{*a}^t], \quad \varphi_{**a}^0 = 0. \quad (2.4)$$

(See Dressler and Farmer [16] for a similar derivation for maps.) Writing out the components on a particular chart explicitly, Eq. (2.4) has the form

$$(\dot{\varphi}_{**a}^t)_{pq}^i = \frac{\partial v^i}{\partial x^k} (\varphi_{**a}^t)_{pq}^k + \frac{\partial^2 v^i}{\partial x^k \partial x^\ell} (\varphi_{*a}^t)_p^k (\varphi_{*a}^t)_q^\ell, \quad (\varphi_{**a}^0)_{pq}^i = 0. \quad (2.5)$$

We call the quadratic form φ_{**a}^t the Hessian of φ^t . The growth rate of the Hessian is characterised by *generalised* or *higher-order* Lyapunov exponents [16,17,11].

These are typically larger in magnitude than the Lyapunov exponents associated with the growth of the tangent map φ_{*a}^t (the Jacobian). This means that numerically integrating Eq. (2.5) directly is even more problematic than for the tangent map. In Ref. [11] extensions of the SVD and *QR* (orthogonal-triangular) decomposition methods were introduced to resolve this numerical difficulty. This allowed the precise verification of the differential constraints predicted in earlier work [8,9].

2.2 The Metric

To measure the norm of vectors, we now introduce a Riemannian metric $(\cdot, \cdot)_x : T_x \mathcal{U} \times T_x \mathcal{U} \rightarrow \mathbb{R}$. This induces a corresponding metric $(\cdot, \cdot)_a^t$ on $T_a \mathcal{U}$ defined by

$$(X, Y)_a^t := (\varphi_{*a}^t \cdot X, \varphi_{*a}^t \cdot Y)_x, \quad (2.6)$$

for all $X, Y \in T_a \mathcal{U}$. At $t = 0$ the two metrics coincide. The reason why we introduce a metric on $T_x \mathcal{U}$ and from it obtain the metric on $T_a \mathcal{U}$, and not the other way around, is best understood in terms of the fluid analogy. We wish to measure the growth of the vector X as it is “dragged” by the flow. Initially, the norm of X is $(X, X)_x^{1/2}$, and at a later time t its norm is $(\varphi_{*a}^t \cdot X, \varphi_{*a}^t \cdot X)_x^{1/2}$. The latter is then used in Eq. (2.6) to define a time-dependent metric directly on $T_a \mathcal{U}$, with no explicit reference to φ^t . Thus, it is the metric on $T_x \mathcal{U}$ that is given.

In component form, we have $(X, Y)_a^t = g_{pq}(t, a) X^p Y^q$, with

$$g_{pq}(t, a) = (\varphi_{*a}^t)_p^i h_{ij}(t, \varphi^t(a)) (\varphi_{*a}^t)_q^j, \quad (2.7)$$

where $h_{ij}(t, \varphi^t(a))$ are the components of the metric on $T_x \mathcal{U}$. In Euclidean space, we have $h_{ij}(t, \varphi^t(a)) = \delta_{ij}$, but we shall retain the more general form. Both g_{pq} and h_{ij} are positive-definite symmetric matrices, since we chose a proper (positive-definite) Riemannian metric.

2.3 A Covariant Time Derivative

To ensure that we are always dealing with tensors—and thus guarantee the covariance of all expressions—it is convenient to introduce a covariant version of the time-derivative operator [21]. In the Eulerian basis, the components of this operator \mathcal{D} acting on a vector $X \in T_x \mathcal{U}$ are

$$\mathcal{D}X^i := \dot{X}^i + \Gamma_{k\ell}^i X^k v^\ell + \gamma_k^i X^k, \quad (2.8)$$

where the Riemann–Christoffel connexion are defined as

$$\Gamma_{ij}^k := \frac{1}{2} h^{k\ell} (h_{i\ell,j} + h_{j\ell,i} - h_{ij,\ell}), \quad (2.9)$$

and

$$\gamma^i_j := \frac{1}{2} h^{i\ell} \frac{\partial h_{\ell j}}{\partial t}. \quad (2.10)$$

We use the symbol ∇_j to denote a covariant derivative with respect to x^j ,

$$\nabla_j X^i := \frac{\partial X^i}{\partial x^j} + \Gamma_{jk}^i X^k. \quad (2.11)$$

Without the γ term, Eq. (2.8) is the usual definition of covariant differentiation along a curve [22], in our case the curve being the trajectory of x with tangent vector v . The extra term allows the metric to depend explicitly on time whilst preserving the compatibility property $\mathcal{D}h_{ij} = 0$. Note that Γ , γ , and v do not transform as tensors, but their combination defined by Eq. (2.8) does [21].

3 Two Orthonormal Frames

In Section 2.1, we mentioned that direct numerical integration of the evolution equation (2.3) for the tangent map φ_{*a}^t is impractical due to the dominance of the eigenvector associated with the direction of fastest stretching. In the present section we use an appropriate matrix decomposition to separates the vastly different scales contained in φ_{*a}^t (Section 3.1), as done by Greene & Kim [14,15], but our treatment includes a metric with explicit time-dependence. We aim to write the equations of motion for the matrices of the decomposition in manifestly covariant form (Section 3.2), using the covariant time derivative of Section 2.3. In addition to clarifying the rôle of a nontrivial metric in chaotic dynamics, the covariant form allows a more straightforward derivation of the equations of motion for the Lagrangian derivatives, as we shall see in Section 4. Finally, in Section 3.3 the finite-time Lyapunov exponents and characteristic directions are defined.

3.1 Decomposition of φ_{*a}^t into Orthonormal Bases

The tangent map φ_{*a}^t is the Jacobian $\partial x/\partial a$ of the transformation from Lagrangian coordinates (a) to Eulerian coordinates (x). Hence, it tells us how an set of orthonormal vectors (with respect to the metric h) at a is transformed to a set of vectors at x . This last set is not orthonormal in general.

We can however decompose φ_{*a}^t into an outer product of orthonormal bases,

$$(\varphi_{*a}^t)_q^i = \sum_{\sigma, \tau} (w_\sigma)^i (e_\tau)_q \eta^{\sigma\tau}. \quad (3.1)$$

where $\eta^{\sigma\tau} = \delta^{\sigma\tau}$, since we have a positive-definite metric, but we write it as η following standard notation [22]. Note that for Greek indices we will *not* use the Einstein sum convention; all sums will be written out explicitly. By “orthonormal,” we mean that the bases w_σ and e_σ satisfy

$$h_{ij}(t, x) = \sum_{\sigma, \tau} (w_\sigma)^i (w_\tau)^j \eta^{\sigma\tau}, \quad g_{pq}(t, a) = \sum_{\sigma, \tau} (e_\sigma)_p (e_\tau)_q \eta^{\sigma\tau}. \quad (3.2)$$

The orthonormal basis vectors e_σ depend on a and t , and the basis vectors w_τ depend on x and t ; they diagonalise g and h , respectively. The basis $\{e_\sigma\}$ is a noncoordinate basis, *i.e.*, it does not correspond to the natural tangent vectors $\{\partial/\partial a^q\}$ of the coordinate system a , or of any other coordinate system. The same is true of the basis $\{w_\sigma\}$. It follows from Eq. (3.2) that

$$(w_\tau)^\ell (w_\sigma)_\ell = \eta_{\tau\sigma}, \quad (e_\tau)^q (e_\sigma)_q = \eta_{\tau\sigma}. \quad (3.3)$$

In the appendix we show that, using the standard singular value decomposition (SVD), we can always write

$$M^i{}_q = \sum_{\sigma, \tau} U^i{}_\sigma F^{\sigma\tau} V_{q\tau}, \quad (3.4)$$

where the matrix M denotes the components of φ_{*a}^t in a particular chart. The notation for the components of the decomposition is that of Refs. [14,15,11]. The matrix F is diagonal, and the matrices U and V satisfy the orthogonality relations,

$$h_{ij} U^i{}_\sigma U^j{}_\tau = \eta_{\sigma\tau}, \quad \eta^{pq} V_{p\sigma} V_{q\tau} = \eta_{\sigma\tau}. \quad (3.5)$$

Note that U is orthogonal with respect to the metric on $T_x \mathcal{U}$, but V is orthogonal with respect to η^{pq} , which is *not* the metric on $T_a \mathcal{U}$. The reason for this is that we want to avoid including in V the exponentially growing terms contained in g .

The decomposition (3.4) is unique up to permutations of rows and columns. The diagonal elements $F^{\sigma\sigma} =: \Lambda^\sigma$ are called the singular values of M . Requiring that the singular values be ordered decreasing in size makes the decomposition unique (for nondegenerate eigenvalues). We refer the reader to Refs. [13,11] for a discussion of the geometrical significance of the SVD in dynamical systems.

From Eqs. (3.1) and (3.4), we can make the identification

$$(e_\sigma)_p = V_{p\rho} F^\rho{}_\sigma, \quad (e_\sigma)^p = \hat{V}^p{}_\rho (F^{-1})^\rho{}_\sigma, \quad (3.6)$$

where we used $\eta_{\mu\nu}$ as a “metric” to raise and lower Greek indices, and we defined $\hat{V}^p{}_\rho := \eta^{pq} V_{q\rho}$ to abridge the notation. Note that $\hat{V}^p{}_\rho$ is a tensor, but is not equal to $V^p{}_\rho = g^{pq} V_{q\rho}$ because η^{pq} is not the metric on $T_a \mathcal{U}$; hence the need for a different symbol.

Analogously to $\{e_\sigma\}$, we can take the basis $\{w_\sigma\}$ to be

$$(w_\sigma)^i = U^i{}_\tau, \quad (w_\sigma)_i = U_{i\tau}. \quad (3.7)$$

With the definitions (3.6) and (3.7) it is easy to check that $\{e_\sigma\}$ and $\{w_\sigma\}$ satisfy the orthonormality conditions (3.2) and (3.3).

3.2 Time Evolution

Greene & Kim [14] derived equations of motion for U , F , and V for a metric without explicit time-dependence. Our derivation parallels theirs, except for the use of the covariant time derivative of Section 2.3 to allow for time-dependence of the metric.

We differentiate the decomposition (3.4) using the covariant derivative (2.8),

$$\mathcal{D}M^i{}_q = 0 = \sum_{\sigma, \tau} [\mathcal{D}U^i{}_\sigma F^{\sigma\tau} V_{q\tau} + U^i{}_\sigma \mathcal{D}F^{\sigma\tau} V_{q\tau} + U^i{}_\sigma F^{\sigma\tau} \mathcal{D}V_{q\tau}], \quad (3.8)$$

and then contract with $U_{i\mu} \hat{V}^q{}_\nu$,

$$\sum_{\sigma, \tau} [U_{i\mu} \mathcal{D}U^i{}_\sigma F^{\sigma\tau} \eta_{\tau\nu} + \eta_{\mu\sigma} \dot{F}^{\sigma\tau} \eta_{\tau\nu} + \eta_{\mu\sigma} F^{\sigma\tau} \hat{V}^q{}_\nu \mathcal{D}V_{q\tau}] = 0, \quad (3.9)$$

where because $F^{\sigma\tau}$ is a scalar we replaced the covariant time derivative by an ordinary one. To simplify the notation, we use the diagonality of F , and raise and lower Greek indices with η , yielding

$$U_{i\mu} \mathcal{D}U^i{}_\nu F^\nu{}_\nu + \dot{F}_{\mu\nu} + F^\mu{}_\mu \hat{V}^q{}_\nu \mathcal{D}V_{q\mu} = 0. \quad (3.10)$$

The scalar $U_{i\mu} \mathcal{D}U^i{}_\nu$ is antisymmetric in μ and ν , but not $\hat{V}^q{}_\nu \mathcal{D}V_{q\mu}$, because $\mathcal{D}\eta^{pq} \neq 0$ since η^{pq} is not the metric. However, the quantity $\hat{V}^q{}_\nu \hat{V}_{q\mu} = \eta^{pq} V_{p\nu} \dot{V}_{q\mu}$ is antisymmetric in μ and ν , so we expand the covariant derivative of V ,

$$U_{i\mu} \mathcal{D}U^i{}_\nu F^\nu{}_\nu + \dot{F}_{\mu\nu} + F^\mu{}_\mu \hat{V}^q{}_\nu \dot{V}_{q\mu} - F^\nu{}_\nu G_{\mu\nu} = 0, \quad (3.11)$$

where

$$G_{\mu\nu} := U^i{}_\mu U^j{}_\nu [\nabla_j v_i + \gamma_{ij}], \quad (3.12)$$

and we used the transformation law from Lagrangian to Eulerian coordinates imposed by the covariance of \mathcal{D} [21]. The matrix G is a covariant generalisation of the velocity gradient tensor, expressed in the U basis. The γ term denotes straining due to the time-dependence of the metric.

We can now solve Eq. (3.11) for the various time derivatives by using the antisymmetry of $U_{\ell\mu} \mathcal{D} U^\ell{}_\nu$ and $\hat{V}^q{}_\mu \dot{V}_{q\nu}$. We obtain finally

$$\dot{F}^\mu{}_\mu = G^\mu{}_\mu F^\mu{}_\mu, \quad (3.13)$$

$$U_{\ell\mu} \mathcal{D} U^\ell{}_\nu = -\frac{G_{\mu\nu} (F^\nu{}_\nu)^2 + G_{\nu\mu} (F^\mu{}_\mu)^2}{(F^\mu{}_\mu)^2 - (F^\nu{}_\nu)^2}, \quad \text{for } \mu \neq \nu; \quad (3.14)$$

$$\hat{V}^q{}_\mu \dot{V}_{q\nu} = -\frac{F^\mu{}_\mu F^\nu{}_\nu}{(F^\mu{}_\mu)^2 - (F^\nu{}_\nu)^2} A_{\mu\nu}, \quad \text{for } \mu \neq \nu; \quad (3.15)$$

where $A_{\mu\nu} := G_{\mu\nu} + G_{\nu\mu}$. These equations are identical in form to those of Greene & Kim [14,15], except for the definition of G and the use of the operator \mathcal{D} instead of the time derivative in the U equation. Note that the matrix A is the rate-of-strain tensor (up to a possible factor of two, depending on the convention used), expressed in the basis U .

The ordinary time derivative is used for F because it is a scalar. For U , we need to use the modified version \mathcal{D} of the time derivative to take into account the nontrivial, possibly time-dependent metric. The ordinary derivative is used for the V equation, even though it is a vector, because we have effectively rescaled the metric g_{pq} to give η_{pq} . The reason for doing so is that for a chaotic flow the elements of g_{pq} grow exponentially, rendering the metric difficult to use directly. Instead, we have absorbed the exponential growth into the $F^\mu{}_\mu$, which are the coefficients of expansion of the flow [2,1].

The techniques used in this section also apply to the QR method for obtaining the Lyapunov exponents and characteristic directions [13,7]. One simply replaces the ordinary time derivative of the Q equation by a covariant derivative (analogous to the U), and the time derivatives of the elements of R are unchanged (analogous to F and V). The $\partial v / \partial x$ term must be modified to use the covariant spatial derivative, and a time derivative of the metric must be included, exactly as in Eq. (3.12). The QR method has the advantage of having no singularity in its equations of motion, as opposed to Eqs. (3.14) and (3.15) which are singular for $\Lambda_\mu = \Lambda_\nu$. However, the type of asymptotic analysis we do here is better done with the SVD method, as it was in Ref. [11].

3.3 Lyapunov Exponents and Characteristic Directions

The relation (3.6) between the orthonormal frame $\{e_\sigma\}$ and the SVD matrices allows the definition of the finite-time Lyapunov exponents

$$\lambda_\sigma(t, a) := \frac{1}{t} \log \Lambda_\sigma(t, a), \quad (3.16)$$

where the $\Lambda_\sigma := F^{\sigma\sigma}$ are called the coefficients of expansion [2]. The Λ_σ give the instantaneous relative growth of the principal axes of an infinitesimal ellipsoid moving with the flow. Taking the limit $t \rightarrow \infty$ in Eq. (3.16) gives the Lyapunov exponents λ_σ^∞ , which are independent of a and t in a given ergodic region [2] (for almost all initial conditions); they converge very slowly [7]. Associated with the coefficients of expansion are the characteristic directions $(\hat{e}_\sigma)_q = V_{q\sigma}$, which converge exponentially rapidly to their time-asymptotic value $\hat{e}_\sigma^\infty(a)$ [7,11] (for nondegenerate Lyapunov exponents). They give the directions of stretching of the axes of the ellipsoid in Lagrangian space.

Following Goldhirsch et al. [7], we shall assume that the Lyapunov exponents λ_σ^∞ are nondegenerate and ordered such that $\Lambda_{\sigma-1} > \Lambda_\sigma$. After allowing some time for chaotic behaviour to set in, we have that $\Lambda_\sigma \gg \Lambda_\kappa$ for $\sigma < \kappa$. Use will be made of this ordering in the next section to estimate the asymptotic behaviour of Lagrangian derivatives.

4 Lagrangian Derivatives

In Thiffeault [11], evolution equations for the Lagrangian derivatives of \hat{e} (V), w (U), and Λ (F) were derived. In this section we show how these equations must be modified to account for a nontrivial metric. We then demonstrate that the asymptotic form for the Lagrangian derivatives, also obtained in Ref. [11], applies to the curved case.

We define the quantities

$$\Psi_{\kappa\nu} := \hat{V}^p{}_\kappa \frac{\partial}{\partial a^p} \log F^\nu{}_\nu, \quad \Phi_{\kappa\mu\nu} := \hat{V}^p{}_\kappa U^i{}_\mu \nabla_p U_{i\nu}, \quad (4.1)$$

$$\Theta_{\kappa\mu\nu} := \hat{V}^p{}_\kappa \hat{V}^q{}_\mu \frac{\partial V_{q\nu}}{\partial a^p} \quad (4.2)$$

which are the Lagrangian derivatives of Λ_ν , $U_{i\nu}$, and $V_{q\nu}$ expressed in the orthonormal frames $\{w_\sigma\}$ and $\{\hat{e}_\sigma\}$. Integration by parts leads to the symmetries $\Phi_{\kappa\mu\nu} = -\Phi_{\kappa\nu\mu}$ and $\Theta_{\kappa\mu\nu} = -\Theta_{\kappa\nu\mu}$. We are using an ordinary spatial

derivative—as opposed to a covariant derivative—in the definition of Θ because we have effectively rescaled the metric g_{pq} to give η_{pq} (Section 2). Using the covariant derivative would destroy the antisymmetry of Θ in its last two indices.

Equations of motion for Ψ , Φ , and Θ can be derived in a manner analogous to Ref. [11], by taking Lagrangian derivatives of the equations of motion (3.13)–(3.15). However, care must be taken when commuting covariant derivatives: since the metric h is nontrivial, curvature terms can arise. Moreover, a form of time-curvature associated with the nontrivial time-dependence of h is also present [21]. The evolution equation for Φ is

$$\begin{aligned}\dot{\Phi}_{\kappa\mu\nu} = & \sum_{\sigma} (V^T \dot{V})^{\sigma}_{\kappa} \Phi_{\sigma\mu\nu} + \sum_{\sigma} (U^T \mathcal{D}U)^{\sigma}_{\mu} \Phi_{\kappa\sigma\nu} - \sum_{\sigma} (U^T \mathcal{D}U)^{\sigma}_{\nu} \Phi_{\kappa\mu\sigma} \\ & + \widehat{V}^q_{\kappa} \frac{\partial}{\partial a^q} (U^T \mathcal{D}U)_{\mu\nu} - \Lambda_{\kappa} (\sum_{\sigma} R_{\mu\nu\kappa\sigma} v^{\sigma} + S_{\mu\nu\kappa})\end{aligned}\quad (4.3)$$

where the matrices $(U^T \mathcal{D}U)$ and $(V^T \dot{V})$ denote the right-hand side of Eqs. (3.14) and (3.15), respectively. The last term in Eq. (4.3) appears only when h has nontrivial spatial and temporal dependence. It contains the curvature R , defined by [22]

$$\sum_{\sigma} R_{\mu\nu\kappa\sigma} v^{\sigma} = U^k_{\mu} U^i_{\nu} U^j_{\kappa} [\nabla_k \nabla_i v_j - \nabla_i \nabla_k v_j] , \quad (4.4)$$

and the time-curvature S [21], defined by

$$S_{\mu\nu\kappa} := U^k_{\mu} U^i_{\nu} U^j_{\kappa} [\nabla_k \gamma_{ji} - \nabla_i \gamma_{jk}] . \quad (4.5)$$

Both $R_{\mu\nu\kappa\sigma}$ and $S_{\mu\nu\kappa}$ are antisymmetric in μ and ν .

Equations of motion can also be found for Ψ and Θ ,

$$\dot{\Psi}_{\kappa\nu} = \sum_{\sigma} (V^T \dot{V})^{\sigma}_{\kappa} \Psi_{\sigma\nu} + \sum_{\sigma} A^{\sigma}_{\nu} \Phi_{\kappa\sigma\nu} + \Lambda_{\kappa} X_{\nu\kappa\nu} , \quad (4.6)$$

$$\begin{aligned}\dot{\Theta}_{\kappa\mu\nu} = & \sum_{\sigma} (V^T \dot{V})^{\sigma}_{\kappa} \Theta_{\sigma\mu\nu} + \sum_{\sigma} (V^T \dot{V})^{\sigma}_{\mu} \Theta_{\kappa\sigma\nu} - \sum_{\sigma} (V^T \dot{V})^{\sigma}_{\nu} \Theta_{\kappa\mu\sigma} \\ & + \widehat{V}^q_{\kappa} \frac{\partial}{\partial a^q} (V^T \dot{V})_{\mu\nu} ,\end{aligned}\quad (4.7)$$

where

$$X_{\nu\kappa\mu} := U^k_{\nu} U^i_{\kappa} U^{\ell}_{\mu} \nabla_i [\nabla_{\ell} v_k + \gamma_{k\ell}] . \quad (4.8)$$

Equations (4.6) and (4.7) are identical in form to their Euclidean versions of Ref. [21], with the proviso that the \widehat{X} in that paper must be replaced by X

of Eq. (4.8). In its new form, which allows for a nontrivial metric, $X_{\nu\kappa\mu}$ is no longer symmetric in κ and μ ; rather, it satisfies the commutation relation

$$X_{\nu\kappa\mu} - X_{\nu\mu\kappa} = \sum_{\sigma} R_{\mu\nu\kappa\sigma} v^{\sigma} + S_{\mu\nu\kappa}. \quad (4.9)$$

Without redoing the entire derivation, we have thus shown that the evolution equations derived in Ref. [11] can be easily modified to allow for a nontrivial metric h . Moreover, it is straightforward to show that these terms do not modify the asymptotic behaviour of the Lagrangian derivatives derived in Ref. [11]. The reason is that if we assume that the motion takes place on some bounded region of phase space, then the curvatures R and S must be bounded.

We now quote the result of Ref. [11], which also applies to our system: for $t \gg 1$, by which we mean that the dynamical system has evolved long enough for the quantities Λ_{μ} to have reached a regime of quasi-exponential behaviour, we have that the Lagrangian derivatives defined by Eqs. (4.1) and (4.2) evolve asymptotically as

$$\Phi_{\kappa\mu\nu} = \max(\Lambda_{\kappa}, \gamma_{\mu\nu}) \tilde{\Phi}_{\kappa\mu\nu}, \quad (4.10)$$

$$\Psi_{\kappa\nu} = \max(\Lambda_{\kappa}, \gamma_{\kappa\kappa}, \gamma_{\nu\nu}) \tilde{\Psi}_{\kappa\nu} + \Psi_{\kappa\nu}^{\infty}, \quad (4.11)$$

$$\Theta_{\kappa\mu\nu} = \max(\gamma_{\mu\nu}\Lambda_{\kappa}, \gamma_{\kappa\kappa}, \gamma_{\mu\mu}, \gamma_{\nu\nu}) \tilde{\Theta}_{\kappa\mu\nu} + \Theta_{\kappa\mu\nu}^{\infty}, \quad (4.12)$$

where

$$\gamma_{\mu\nu} := \begin{cases} \Lambda_{\nu}/\Lambda_{\mu}, & \mu < \nu; \\ \Lambda_{\mu}/\Lambda_{\nu}, & \mu > \nu; \\ \max\left(\frac{\Lambda_{\mu}}{\Lambda_{\mu-1}}, \frac{\Lambda_{\mu+1}}{\Lambda_{\mu}}\right), & \mu = \nu. \end{cases} \quad (4.13)$$

The quantities with tildes have algebraic (nonexponential) behaviour, and the quantities with $^{\infty}$ superscripts are constants. The γ 's are defined such that asymptotically they are decreasing exponentially, for nondegenerate Λ . See Ref. [11] for a more detailed discussion of the character of this asymptotic behaviour.

5 Constraints

The Hessian φ_{**a}^t , introduced in Section 2, is the tensor of second derivatives of $\varphi^t(a)$; it characterises deformations of fluid elements beyond ellipsoidal. Because partial derivatives commute, the Hessian is symmetric in the corresponding indices. In Ref. [11] we showed that this symmetry of the Hessian implies that the Lagrangian derivatives of Section 4 are not all independent.

However, the relationship between the derivatives is very singular for chaotic flows. We show in this section that the relations can be used to obtain constraints on the derivatives, recovering and extending previous results [8,10,9] deduced by different means.

5.1 Symmetry of the Hessian

We define the projection of the Hessian $(\varphi_{**a}^t)_{pq}^\ell = \partial^2 x^\ell / \partial a^p \partial a^q$ onto the U and V bases as

$$K_{\kappa\mu\nu} := U_{\ell\kappa} (\varphi_{**a}^t)_{pq}^\ell \hat{V}^p_\mu \hat{V}^q_\nu = K_{\kappa\nu\mu}. \quad (5.1)$$

Since $(\varphi_{*a}^t)_q^\ell = \partial x^\ell / \partial a^q$, we can write

$$\frac{\partial^2 x^\ell}{\partial a^p \partial a^q} = \frac{\partial(\varphi_{*a}^t)_q^\ell}{\partial a^p}, \quad (5.2)$$

and using the decomposition (3.4) for $M^i{}_q = (\varphi_{*a}^t)_q^i$, we find

$$K_{\kappa\mu\nu} = \Lambda_\kappa \Psi_{\mu\kappa} \eta_{\nu\kappa} + \Lambda_\kappa \Theta_{\mu\nu\kappa} + \Lambda_\nu \Phi_{\mu\kappa\nu}. \quad (5.3)$$

The Hessian is symmetric in its lower indices, so we can equally well write

$$K_{\kappa\mu\nu} = \Lambda_\kappa \Psi_{\nu\kappa} \eta_{\mu\kappa} + \Lambda_\kappa \Theta_{\nu\mu\kappa} + \Lambda_\mu \Phi_{\nu\kappa\mu}, \quad (5.4)$$

where we interchanged μ and ν in (5.3). Equating (5.3) and (5.4), we find the relations

$$\Lambda_\mu (\Theta_{\mu\mu\nu} + \eta_{\mu\mu} \Psi_{\nu\mu}) = \Lambda_\nu \Phi_{\mu\mu\nu}, \quad \mu \neq \nu, \quad (5.5)$$

$$\Lambda_\kappa (\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa}) = \Lambda_\nu \Phi_{\mu\nu\kappa} - \Lambda_\mu \Phi_{\nu\mu\kappa}, \quad \mu, \nu, \kappa \text{ differ.} \quad (5.6)$$

Equations (5.5) and (5.6) can be used to solve for the Θ 's in terms of the Φ 's and Ψ 's. However, for chaotic flows the exponential behaviour of the Λ 's renders this solution highly singular [11]. Thus, in practise it is better to evolve all the Lagrangian derivatives.

Rather than solving for the Θ , if the flow is chaotic the relations (5.5) and (5.6) can be put to good use in another manner. The Lagrangian derivatives of U , as contained in Φ , are not quantities of great interest to us. They contain information about the sensitive dependence on initial conditions of the absolute orientation of fluid elements in Eulerian space. This information is not necessary for solving problems in Lagrangian coordinates, and is too sensitive to initial conditions to be of use anyhow. We thus substitute the time-asymptotic form of Φ , given by Eq. (4.10), in the right-hand side of Eqs. (5.5) and (5.6),

yielding

$$\Theta_{\mu\mu\nu} + \eta_{\mu\mu}\Psi_{\nu\mu} = \max\left(\Lambda_\nu, \frac{\Lambda_\nu}{\Lambda_\mu} \gamma_{\mu\nu}\right) \tilde{\Phi}_{\mu\mu\nu}, \quad (5.7)$$

$$\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa} = \frac{\Lambda_\nu}{\Lambda_\kappa} \max(\Lambda_\mu, \gamma_{\nu\kappa}) \tilde{\Phi}_{\mu\nu\kappa} - \frac{\Lambda_\mu}{\Lambda_\kappa} \max(\Lambda_\nu, \gamma_{\mu\kappa}) \tilde{\Phi}_{\nu\mu\kappa}, \quad (5.8)$$

where, as before, μ , ν , and κ differ.

5.2 Type I Constraints

For $\mu < \nu$, Eq. (5.7) can be written

$$\Theta_{\mu\mu\nu} + \Psi_{\nu\mu} = \max\left(\Lambda_\nu, \gamma_{\mu\nu}^2\right) \tilde{\Phi}_{\mu\mu\nu}, \quad \mu < \nu, \quad (5.9)$$

where we have set $\eta_{\mu\mu} = 1$ for a proper Riemannian metric. If the index ν corresponds to a contracting direction ($\Lambda_\nu \ll 1$), the right-hand side of Eq. (5.9) goes to zero exponentially fast, at a rate Λ_ν or $\gamma_{\mu\nu}^2$, whichever is slowest. In that case Eq. (5.9) is a *constraint* implying that for large t we have

$$(\Theta_{\mu\mu\nu} + \Psi_{\nu\mu}) \longrightarrow 0, \quad \Lambda_\nu \ll \Lambda_\mu, \quad \Lambda_\nu \ll 1. \quad (5.10)$$

We refer to (5.10) as Type I constraints. The total number of such Type I constraints is

$$N_I = n_s \left[n - \frac{1}{2}(n_s + 1) \right], \quad (5.11)$$

where n is the dimension of the space and n_s is the number of contracting directions (i.e., the number of negative Lyapunov exponents) possessed by the flow in a particular chaotic region. Table 1 gives the number of Type I constraints, N_I , as a function of n and n_s .

In two dimensions, we typically have one contracting direction, so there is a single Type I constraint. This is the same constraint that was derived in Refs. [8,9].

In three dimensions, for an autonomous flow, we typically also have one contracting direction. There are then two Type I constraint. These constraints correspond to those derived in Ref. [9].

An interesting special case of the Type I constraints is obtained by setting $\nu = n$ in Eq. (5.9), and then summing over $\mu < n$. We obtain

$$\nabla_q (\hat{e}_n)^q - (\hat{e}_n)^q \nabla_q \log \Lambda_n \sim \max\left(\Lambda_n, \gamma_{nn}^2\right) \rightarrow 0. \quad (5.12)$$

Table 1

The total number of Type I constraints for low-dimensional systems, as given by Eq. (5.11). The rows denote n , the columns $n_s \leq n$.

n	1	2	3	4	5	6	7
1	0						
2	1	1					
3	2	3	3				
4	3	5	6	6			
5	4	7	9	10	10		
6	5	9	12	14	15	15	
7	6	11	15	18	20	21	21

This constraint was discovered numerically and used by Tang and Boozer [8,10,23,24] and derived in three dimensions by Thiffeault and Boozer [9]. The present method not only gives the constraint in a very direct manner for any dimension n , but it also provides us with its asymptotic convergence rate, as determined by the right-hand side of (5.12).

5.3 Type II Constraints

Equation (5.14) implies that

$$\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa} \sim \max\left(\frac{\Lambda_\mu \Lambda_\nu}{\Lambda_\kappa}, \frac{\Lambda_\nu}{\Lambda_\kappa} \gamma_{\nu\kappa}, \frac{\Lambda_\mu}{\Lambda_\kappa} \gamma_{\mu\kappa}\right). \quad (5.13)$$

We are interested in finding constraints analogous to the Type I constraints of Section 5.2. It is clear that unless both μ and ν are greater than κ , the right-hand side of Eq. (5.13) is of order unity or greater, and so does not go to zero. We can assume without loss of generality that $\mu < \nu$, so that

$$\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa} \sim \gamma_{\mu\kappa} \max(\Lambda_\nu, \gamma_{\mu\kappa}), \quad \nu > \mu > \kappa, \quad (5.14)$$

where we have used $\gamma_{\nu\kappa} \ll \gamma_{\mu\kappa}$. Whether or not Eq. (5.14) is a constraint depends on the specific behaviour of $\Lambda_\mu \Lambda_\nu / \Lambda_\kappa$. Clearly, we have a constraint if $\Lambda_\nu \ll 1$, since $\gamma_{\mu\kappa} \ll 1$. This provides a lower bound on the number N_{II} of Type II constraints; by choosing ν from the n_s contracting directions, and summing over the remaining $\kappa > \mu > \nu$, we obtain

$$N_{\text{II}} \geq \frac{1}{2} n_s \left[n^2 - (n_s + 2) \left(n - \frac{1}{3}(n_s + 1) \right) \right]. \quad (5.15)$$

Table 2

Lower bound on the number of Type II constraints, as given by Eq. (5.15). The rows denote n , the columns $n_s \leq n$.

n	1	2	3	4	5	6	7
1	0						
2	0	0					
3	1	1	1				
4	3	4	4	4			
5	6	9	10	10	10		
6	10	16	19	20	20	20	
7	15	25	31	34	35	35	35

But even if $\Lambda_\nu \gg 1$ we can have a constraint, as long as $\Lambda_\nu \gamma_{\mu\kappa} \ll 1$. This depends on the particular problem at hand; hence, Eq. (5.15) is only a lower bound, but a fairly tight one for low dimensions. Table 2 enumerates the minimum number of Type II constraints as a function of n and n_s .

Note that when κ , μ , and ν differ we can write

$$\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa} = -(\hat{e}_\kappa)_q [\hat{e}_\mu, \hat{e}_\nu]^q, \quad (5.16)$$

where the Lie bracket is

$$[\hat{e}_\mu, \hat{e}_\nu]^q := (\hat{e}_\mu)^p \frac{\partial}{\partial a^p} (\hat{e}_\nu)^q - (\hat{e}_\nu)^p \frac{\partial}{\partial a^p} (\hat{e}_\mu)^q. \quad (5.17)$$

The Type II constraints are thus forcing certain Lie brackets of the characteristic directions \hat{e}_σ to vanish asymptotically. The geometrical implications of this, and perhaps a connexion to the Frobenius theorem and the existence of submanifolds, remains to be explored.

6 Curvature

We now make contact with the earlier results of Refs. [8,9], where the constraints were derived for two and three dimensional flows by examining the form of the Riemann curvature tensor associated with the metric g . We list the advantages of the present method. The only known disadvantage is that the formalism presented here does not apply to maps, only to continuous flows. This is because of the lack of an SVD method for maps [13]. The use of the

QR method, which is defined for maps [1,13], may circumvent that problem, but the analysis is more difficult and has not been carried out yet.

Nontrivial metrics can have curvature, as reflected by the presence of the Riemann curvature tensor in Section 4. A straightforward method of obtaining that tensor is through the use of *Ricci rotation coefficients* [22],

$$\omega_{\kappa\mu\nu} := (w_\kappa)^i (w_\mu)^j \nabla_i (w_\nu)_j = (e_\kappa)^p (e_\mu)^q \nabla_p (e_\nu)_q. \quad (6.1)$$

These satisfy the antisymmetry property $\omega_{\kappa\mu\nu} = -\omega_{\kappa\nu\mu}$, and can be rewritten in terms of the Φ of Eq. (4.1) as

$$\omega_{\kappa\mu\nu} = \Lambda_\kappa^{-1} \Phi_{\kappa\mu\nu}. \quad (6.2)$$

In terms of the rotation coefficients, the Riemann curvature tensor is [22, p. 51]

$$R_{\mu\nu\kappa\sigma} = (e_\kappa)^q \nabla_q \omega_{\sigma\mu\nu} - (e_\sigma)^q \nabla_q \omega_{\kappa\mu\nu} - \eta^{\tau\rho} [\omega_{\kappa\rho\mu} \omega_{\sigma\tau\nu} - \omega_{\sigma\rho\mu} \omega_{\kappa\tau\nu} + \omega_{\kappa\rho\sigma} \omega_{\tau\mu\nu} - \omega_{\sigma\rho\kappa} \omega_{\tau\mu\nu}]. \quad (6.3)$$

If we use the relations (5.5) and (5.6) to solve for Φ in terms of Θ and Ψ , we can rewrite Eq. (6.2) as

$$\begin{aligned} \omega_{\kappa\mu\nu} &= \frac{1}{2\Lambda_\kappa\Lambda_\mu\Lambda_\nu} \left\{ \Lambda_\mu^2 (\Theta_{\nu\kappa\mu} - \Theta_{\kappa\nu\mu}) + \Lambda_\nu^2 (\Theta_{\kappa\mu\nu} - \Theta_{\mu\kappa\nu}) - \Lambda_\kappa^2 (\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa}) \right\} \\ &\quad + \frac{1}{\Lambda_\nu} \eta_{\mu\kappa} \Psi_{\nu\mu} - \frac{1}{\Lambda_\mu} \eta_{\nu\kappa} \Psi_{\mu\nu} \end{aligned} \quad (6.4)$$

The form of the curvature obtained by inserting Eq. (6.4) into (6.3) is essentially the one obtained in three dimensions in Ref. [9]. The curvature was calculated directly from the Christoffel symbols in Ref. [8]. The constraints were then deduced by imposing the finiteness of the curvature tensor: some terms in Eq. (6.4) would appear to grow exponentially, so their coefficient must go to zero to maintain the curvature finite. In this manner, the Type I and Type II constraints were derived in two and three dimensions, backed by numerical evidence [8,9,11].

The approach used in this paper to derive the constraints is advantageous in several ways: (i) It is valid in any number of dimensions; (ii) It avoids using the curvature; (iii) There are no assumptions about the growth rate of individual terms in the curvature [9]; (iv) The convergence rate of the constraints is rigorously obtained [Eqs. (5.9) and (5.14)]; (v) The number of constraints can be predicted [Eqs. (5.11) and (5.15)]. The crux of the difference between the two approaches is that here we use the variable Φ to estimate the asymptotic behaviour of the constraints directly, rather than relying on indirect evidence from the curvature. Thus, the derivation of the time-asymptotic form of Φ is essential (Section 4 and Ref. [11]).

7 Discussion

In this paper we have demonstrated that in chaotic flows the Lagrangian derivatives of finite-time Lyapunov exponents, and their associated characteristic directions, satisfy time-asymptotic differential constraints. The constraints are valid for any metric tensor, and are satisfied with quasi-exponential (that is, exponential in the long-time limit) accuracy in time. The method employed to derive the constraints made use of earlier work on the asymptotic form of Lagrangian derivatives [11]. The new results generalise previous work [8,9] by providing constraints in any number of dimensions, together with their convergence rate. We now give a few examples of possible applications of the constraints, and directions for future research.

In Ref. [8] it was demonstrated that for the isotropic, homogeneous advection–diffusion equation the diffusion proceeds almost exclusively along the characteristic direction associated with the smallest Lyapunov exponents. This was shown by transforming the equation to Lagrangian coordinates, thereby eliminating the advection term. The resulting equation was an anisotropic diffusion equation, where the diffusion tensor was proportional to g^{-1} , the inverse of the metric tensor in Lagrangian space. Because the inverse of the metric appears, the dominant eigendirection is that associated with the *smallest* Lyapunov exponent. Physically, this means that diffusion occurs in the direction along which large gradients are created by the stretching effect of the flow.

If the diffusion tensor in the advection–diffusion equation is anisotropic and inhomogeneous, then the direction of fastest diffusion is associated with the characteristic direction calculated using the diffusion tensor itself as a metric. Furthermore, in the large Péclet number limit (small diffusivity), the Type I constraints derived in Section 5.2 can actually be used to transform the advection–diffusion equation into a scalar one-dimensional diffusion equation. This use of the constraints will be addressed in subsequent work [19]. Other applications of the Type I constraint can be found in Tang and Boozer [23,24]: they demonstrate that regions of high curvature of the integral curves of \hat{e}_n lead to locally small finite-time Lyapunov exponents, hindering the chaotic enhancement of diffusion.

In three dimensions with an Euclidean metric, the Type II constraint (5.14) can be written

$$\hat{e}_1 \cdot \nabla \times \hat{e}_1 \sim \gamma_{12} \max(\Lambda_3, \gamma_{12}) \rightarrow 0, \quad (7.1)$$

where the curl is taken with respect to Lagrangian coordinates, a . Equation (7.1) implies that, asymptotically, \hat{e}_1 is of the form

$$\hat{e}_1 = \nabla\psi / \|\nabla\psi\|, \quad (7.2)$$

where $\|\cdot\|$ is the Euclidean norm, and ψ is a pseudopotential function. Equation (7.2) is valid locally. Hence, the field of unstable directions (associated with the largest Lyapunov exponent) has a simple asymptotic structure. The implications of that structure remain to be investigated. An indication of how it might prove useful was investigated in Ref. [9], where it was shown that the Type II constraints could have an impact on energy dissipation in the fast kinematic dynamo.

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A Singular Value Decomposition with a Metric

The decomposition given by Eq. (3.4) is defined for a space with a nontrivial metric, but the numerical algorithms for obtaining the SVD are based on “true” orthogonal matrices (that is, matrices orthogonal with respect to the Euclidean metric). In this appendix we show that we can always find the bases $\{e_\sigma\}$ and $\{w_\sigma\}$ using the ordinary singular value decomposition of matrices with a Euclidean metric.

Since the matrix h_{ij} of components of the metric $(\cdot, \cdot)_x$ on $T_x \mathcal{U}$ is symmetric, it can be diagonalised by an orthogonal transformation. This transformation can then be composed with a rescaling to make the diagonal elements equal to unity (a proper Riemannian metric has positive eigenvalues), yielding

$$h_{ij} = k^{i'}_i \eta_{i'j'} k^{j'}_j, \quad (\text{A.1})$$

where $k^{i'}_i(t, x)$ is the coordinate transformation, and $\eta_{i'j'} = \delta_{i'j'}$ for a positive-definite metric. We are using primed indices to denote this new basis on $T_x \mathcal{U}$ where h is diagonal. The metric g of Eq. (2.7) can then be written

$$g_{pq} = (\varphi_{*a}^t)_p^i k^{i'}_i \eta_{i'j'} k^{j'}_j (\varphi_{*a}^t)_q^j. \quad (\text{A.2})$$

Define the matrix \mathcal{M} with components

$$\mathcal{M}^{i'}_q := k^{i'}_k (\varphi_{*a}^t)_q^k, \quad (\text{A.3})$$

so that Eq. (A.2) can be written

$$g_{pq} = \mathcal{M}^{i'}_p \eta_{i'j'} \mathcal{M}^{j'}_q. \quad (\text{A.4})$$

Since $\eta_{i'j'} = \delta_{i'j'}$, the metric is now in a form $g = \mathcal{M}^T \mathcal{M}$, suggesting the use of the usual SVD technique to eliminate the Eulerian information in g . We have absorbed the non-Euclidean metric into \mathcal{M} , so that even though the space is not Euclidean the decomposition carries through in the same manner as in Refs. [14,11].

Proceeding with the singular value decomposition, we write

$$\mathcal{M}^{i'}_q = Q^{i'}_\sigma F^{\sigma\tau} V_{q\tau}, \quad (\text{A.5})$$

where Q and V are ordinary orthogonal matrices (*i.e.*, with respect to the Euclidean metric) and F is diagonal.

The basis $\{e_\sigma\}$ is given by V and F as in Eq. (3.6). The basis $\{w_\sigma\}$ is given in terms of Q and k by

$$(w_\sigma)^i = U^i_\sigma = (k^{-1})_{i'}^{i'} Q^{i'}_\sigma, \quad (w_\sigma)_i = U_{i\sigma} = \eta_{i'j'} k^{i'}_i Q^{j'}_\sigma. \quad (\text{A.6})$$

Since the SVD always exists, we have demonstrated that the construction of the basis vectors e_σ and w_σ satisfying Eqs. (3.1) and (3.2) is possible.